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On rough weighted ideal convergence of triple sequence of Bernstein polynomials

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Abstract. We introduce and study some basic properties of rough $I_\lambda-$convergent of weight g, where $g : \mathbb{N}^3 \to [0,\infty)$ is a function satisfying $g(m,n,k) \to \infty$ and $\frac{|(m,n,k)|}{g(m,n,k)} \not\to 0$ as $m,n,k \to \infty$, of triple sequence of Bernstein polynomials and also investigate certain properties of rough $I_\lambda-$convergence of weight g.

Keywords: triple sequences, rough convergence, closed and convex, cluster points and rough limit points, Bernstein polynomials, $I-$statistical convergence of order g.

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1. Introduction

Kostyrko et al. [15] and Nuray and Ruckle [18] independently studied in details about the notion of ideal convergence which is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. Later on it was further investigated by many authors, e.g. Šalát et al [25], Hazarika and Mohiuddine [14], and references therein.
Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if and only if (i) $\phi \in I$. (ii) I is additive under union (iii) hereditary. An ideal $I \subseteq P(S)$ is said to be *non trivial* if $I \neq \phi$ and $S \notin I$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if and only if (i) $\phi \notin F$ (ii) for each $A, B \in F$ we have $A \cap B \in F$ (iii) for each $A \in F$ and $B \supset A$, implies $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. We say that a non-trivial ideal $I \subseteq P(S)$ is (i) an *admissible ideal* on S if and only if it contains all singletons, i.e. if it contains $\{\{x\} : x \in S\}$ (ii) *maximal*, if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset (iii) is said to be a *translation invariant ideal* if $\{k+1 : k \in A\} \in I$, for any $A \in I$. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ (denoted by $I\text{-}\lim x_k = \ell$) if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N}, : |x_k - \ell| \geq \varepsilon\} \in I$.

The idea of rough convergence was first introduced by Phu [20, 21, 22] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and

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convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r. Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained to statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$limit set of the sequence is equal to intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. Dündar and Cakan [6] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. Dündar [8] introduced rough ideal convergence for double sequences. In [24], Sahiner and Tripathy introduced the notion of $I-$convergence of a triple sequences, which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence. The different types of notions of triple sequence was introduced and investigated by Sahiner et al. [23]. Later on further studied by Esi [9, 13] , Esi and Catalbas [10], Esi and Şavas [11], Esi et al. [12], Dutta et al. [4], Debnath et al. [5], Malik and Maity [16], Pal et al. [19], Şavas and Esi [26], Tripathy and Goswami [29] and many others.

Given an increasing function $\phi:\mathbb{N}\to(0,\infty)$ with $\lim_{n\to\infty}\phi(n)=\infty$, Niculescu and Prajitura [17] define the upper density of weight ϕ by the formula

$$\bar{d}_\phi(A)=\limsup_{n\to\infty}\frac{|A\cap[1,n]|}{\phi(n)},$$

where $A\subset\mathbb{N}$, and $|.|$ denote the cardinality of a set.
Let $\omega=\{0,1,2,....\}$, and a function $g:\omega\to[0,\infty)$, where $\lim_{n\to\infty}g(n)=\infty$, and $\frac{n}{g(n)}\nrightarrow 0$ as $n\to\infty$. Denote G the set of all such functions g. Balcerzak et al. [3] define the upper density of weight g by the formula

$$\bar{d}_g(A)=\limsup_{n\to\infty}\frac{\mathrm{card}(A\cap n)}{g(n)}\text{ for }A\subset\omega.$$

Consider the family $\mathcal{Z}_g=\{A\subset\omega:\bar{d}_g(A)=0\}$. Obviously $\mathcal{Z}_g$ is an ideal of ω.
Based on this idea, recently Şavas [27] introduced I_λ-statistical convergence of weight g for real sequence and proved some interesting results.
In this paper we investigate some basic properties of rough wighted $I-$convergence of a triple sequence of Bernstein polynomials in three dimensional cases which are not earlier. We study the set of all rough wighted $I-$limits of a triple sequence of Bernstein polynomials and also the relation between analyticness and rough wighted $I-$convergence of a triple sequence of Bernstein polynomials.

Let U be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $U_{ik\ell}=\{(m,n,k)\in U: m\geq i, n\leq j, k\leq \ell\}$. Then the natural density of U is given by

$$\delta_3(U)=\lim_{i,j,\ell\to\infty}\frac{|U_{ij\ell}|}{ij\ell},$$

where $|U_{ij\ell}|$ denotes the number of elements in $U_{ij\ell}$.

In [28], Stancu introduced the polynomials of Bernstein type of two variables.
For a given continuous function f defined on $D=[0,1]\times[0,1]\times[0,1]$. The Bernstein

polynomials of three variables defined on $C(D)$ by

$$\bar{B} = B_{mnk}\left(f; x, y, z\right) = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{k} a_{m,r}(x) b_{n,s}(y) c_{k,t}(z) f\left(\frac{r}{m}, \frac{s}{n}, \frac{t}{k}\right),$$

where

$$a_{m,r}(x) = \binom{m}{r} x^r \left(1 - x\right)^{m-r};$$

$$b_{n,s}(y) = \binom{n}{s} y^s (1 - y)^{n-s}$$

and

$$c_{k,t}(z) = \binom{k}{t} z^t (1 - z)^{k-t}.$$

Throughout the paper, $\mathbb{R}^3$ denotes the real of three dimensional space with usual metric. Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f; x, y, z))$ such that $(B_{mnk}(f; x, y, z))$ belong to $\mathbb{R}^3$, for $m, n, k \in \mathbb{N}$.
Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f; x, y, z))$ is said to be statistically convergent to $f(x, y, z) \in \mathbb{R}^3$, written as $st - \lim B_{mnk}(f; x, y, z) = f(x, y, z)$, provided that the set

$$U_\epsilon := \left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f; x, y, z) - f(x, y, z)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta_3(U_\epsilon) = 0$. That is,

$$\lim_{rst\to\infty} \tfrac{1}{rst} \left|\{(m, n, k) \leq (r, s, t) : |B_{mnk}(f; x, y, z) - f(x, y, z)| \geq \epsilon\}\right| = 0.$$

In this case, we write $st_3 - \lim B_{mnk}(f; x, y, z) = f(x, y, z)$ or $B_{mnk}(f; x, y, z) \xrightarrow{st_3} f(x, y, z)$.

Throughout the paper we denote χ_A−the characteristic function of $A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. A subset A of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have asymptotic density $d_3(A)$ if

$$d_3(A) = \lim_{ij\ell\to\infty} \tfrac{1}{ij\ell} \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} \chi_A(x).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{C}$ denote the set of complex numbers.

2. Definitions and Preliminaries

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3; m, n, k \in \mathbb{N}$. Also I is an admissible ideal of $2^{\mathbb{N}^3}$. The following definition are obtained:

Definition 2.1. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f; x, y, z))$ is said to be statistically convergent to $f(x, y, z)$ denoted by $B_{mnk}(f; x, y, z) \xrightarrow{st_3} f(x, y, z)$, provided that the set*

$$\left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f; x, y, z) - f(x, y, z)| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$.
In this case, $f(x, y, z)$ is called the statistical limit of the sequence of Berstein polynomials.

Definition 2.2. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ in a metric space $(\mathbb{R}^3, |.,.|)$ and r be a non-negative real number is said to be $r-$convergent to $f(x,y,z)$, denoted by $B_{mnk}(f;x,y,z) \xrightarrow{r} f(x,y,z)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $m,n,k \geq N_\epsilon$ we have*

$$|B_{mnk}(f;x,y,z) - f(x,y,z)| < r + \epsilon$$

In this case $f(x,y,z)$ is called an $r-$limit of $B_{mnk}(f;x,y,z)$.

Remark 2.3. *We consider $r-$limit set of $B_{mnk}(f;x,y,z)$ which is denoted by $LIM^r_{\bar{B}}$ and is defined by*

$$LIM^r_{\bar{B}} = \left\{ B_{mnk}(f;x,y,z) \in \mathbb{R}^3 : B_{mnk}(f;x,y,z) \xrightarrow{r} f(x,y,z) \right\}.$$

Definition 2.4. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is said to be $r-$convergent if $LIM^r_{\bar{B}} \neq \phi$ and r is called a rough convergence degree of $B_{mnk}(f;x,y,z)$. If $r = 0$ then it is ordinary convergence of triple sequence of Bernstein polynomials.*

Definition 2.5. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ in a metric space $(\mathbb{R}^3, |.,.|)$ and r be a non-negative real number is said to be $r-$statistically convergent to $f(x,y,z)$, denoted by $B_{mnk}(f;x,y,z) \xrightarrow{r-st_3} f(x,y,z)$, if for any $\epsilon > 0$ we have $d_3(A(\epsilon)) = 0$, where*

$$A(\epsilon) = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}.$$

In this case $f(x,y,z)$ is called $r-$statistical limit of $B_{mnk}(f;x,y,z)$. If $r = 0$ then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

3. Rough $I_\lambda-$convergence

Definition 3.1. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ in a metric space $(\mathbb{R}^3, |.,.|)$ and r be a non-negative real number, is said to be rough λ-ideal convergent of weight g or $(rI_\lambda)^g-$convergent to $f(x,y,z)$, denoted by $B_{mnk} \xrightarrow{(rI_\lambda)^g} f(x,y,z)$, if for any $\epsilon > 0$ we have*

$$\left\{ p,q,j \in \mathbb{N}^3 : \tfrac{1}{g(\lambda_{pqj})} |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon \right\} \in I.$$

In this case $f(x,y,z)$ is called $rI_\lambda-$limit of $(B_{mnk}(f;x,y,z))$ and a triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is called rough $I_\lambda-$convergent weight g to $f(x,y,z)$ with r as roughness of degree. If $r = 0$ then it is ordinary $I_\lambda-$convergent of weight g. We denote $(rI_\lambda)^g$ the set of all rough λ-ideal convergent of weight g of triple sequence of Bernstein polynomials.

Note 3.2. *It is clear that $rI^g_\lambda-$limit is not necessarily unique.*

Definition 3.3. *Consider $rI^g_\lambda-$limit set of $B_{mnk}(f;x,y,z)$, which is denoted by*

$$I^g_\lambda - LIM^r_{\bar{B}} = \left\{ f(x,y,z) \in \mathbb{R}^3 : B_{mnk}(f;x,y,z) \xrightarrow{(rI_\lambda)^g} f(x,y,z) \right\},$$

then the triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is said to be $rI_\lambda-$convergent of weight g, if $I^g_\lambda - LIM^r_{\bar{B}} \neq \phi$ and r is called degree of rough $I_\lambda-$convergence of weight g of $B_{mnk}(f;x,y,z)$.

Let $\lambda = (\lambda_{pqj})_{p,q,j\in\mathbb{N}}$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{(pqj)+1} \leq \lambda_{pqj} + 1, \lambda_{111} = 1.$$

The collection of such sequences λ will be denoted by η.
We define the generalized de la Valée-Poussin mean of weight g by

$$t_{pqj}(x) = \frac{1}{g(\lambda_{pqj})} \sum_{(m,n,k)\in I_{pqj}} x_{mnk}.$$

where $I_{rst} = [(pqj) - \lambda_{pqj+1}, pqj]$.

Definition 3.4. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is said to be $[V,\lambda](I)^g$ −summable to $f(x,y,z)$, if*

$$I - \lim_{pqj} t_{pqj}(B_{mnk}(f;x,y,z)) = f(x,y,z).$$

i.e., for any $\epsilon > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : |t_{pqj}(B_{mnk}(f;x,y,z)) - f(x,y,z)| \geq \epsilon\right\} \in I$$

and it is denoted by $[V,\lambda](I)^g$.

Definition 3.5. *et f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is said to be rough I_λ−statistically convergent of weight g, if for every $\epsilon > 0$ and $\delta > 0$ the set*

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}| \geq \delta\right\}$$

*belong to I. In this case we write $(rS(I_\lambda))^g - \lim B_{mnk}(f;x,y,z) = f(x,y,z)$.
Or $B_{mnk}(f;x,y,z) \xrightarrow{(rS(I_\lambda))^g} f(x,y,z)$.*

Definition 3.6. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials $(B_{mnk}(f;x,y,z))$ is said to be $[V,\lambda](rI)^g$ −summable to $f(x,y,z)$, if for any $\epsilon > 0$,*

$$\left\{(p,q,j) \in \mathbb{N}^3 : |t_{pqj}(B_{mnk}(f;x,y,z)) - f(x,y,z)| \geq r + \epsilon\right\} \in I$$

and it is denoted by $[V,\lambda](rI)^g$.

Theorem 3.7. *Let f be a continuous function defined on D. A triple sequence of Bernstein polynomials of $(B_{mnk}(f;x,y,z))$ of real numbers and $g_1, g_2 \in G$ be such that there exist $M > 0$ and $u_0, v_0, w_0 \in \mathbb{N}$ such that $\frac{g_1(\lambda_{pqj})}{g_2(\lambda_{pqj})} \leq M$ for all $(p,q,j) \geq (u_0,v_0,w_0)$. Then $(rS(I_\lambda))^{g_1} \subset (rS(I_\lambda))^{g_2}$.*

Proof. For any $\epsilon > 0$,

$$\begin{aligned}
&\frac{|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}|}{g_2(\lambda_{pqj})} \\
&= \frac{g_1(\lambda_{pqj}) |\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}|}{g_2(\lambda_{pqj}) g_1(\lambda_{pqj})} \\
&\leq M \frac{|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}|}{g_1(\lambda_{pqj})}
\end{aligned}$$

for $(p,q,j) \geq (u_0, v_0, w_0)$. Thus for any $\delta > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_2(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_1(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq \frac{\delta}{M}\right\}$$
$$\cup \{1,2,\cdots,(u_0,v_0,w_0)\}.$$

If $(B_{mnk}(f;x,y,z)) \in (rS(I_\lambda))^{g_1}$, then from the above result we get $(rS(I_\lambda))^{g_1} \subset (rS(I_\lambda))^{g_2}$. □

Theorem 3.8. *Let $(B_{mnk}(f;x,y,z))$ be a triple sequence of Bernstein polynomials of real numbers. Then $(rS(I))^g \subset (rS(I_\lambda))^g$ if $\liminf_{pqj} \frac{g(\lambda_{pqj})}{g(pqj)} > 0$.*

Proof. Given that $\liminf_{pqj} \frac{g(\lambda_{pqj})}{g(pqj)} > 0$, then we can find a $M > 0$ such that for sufficiently large (p,q,j), $\frac{g(\lambda_{pqj})}{g(pqj)} \geq M$.
Suppose $B_{mnk}(f;x,y,z) \xrightarrow{(rS(I))^g} f(x,y,z)$, hence for every $\epsilon > 0$ and sufficiently large (r,s,t),

$$\frac{1}{g(pqj)} \left|\{(m,n,k) \leq (p,q,j) : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$\geq \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$\geq M \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|.$$

Then for any $\delta > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(pqj)} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq M\delta\right\}$$
$\in I$, since I is admissible ideal

The result follows from the above inclusion. □

Theorem 3.9. *Let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers. If $\lambda \in \eta$ be such that $\lim_{pqj} \frac{(pqj)-\lambda_{pqj}}{g(pqj)} = 0$, then $(rS(I_\lambda))^g \subset (rS(I))^g$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{pqj} \frac{(pqj)-\lambda_{pqj}}{g(pqj)} = 0$, so we can choose $(u, v, w) \in \mathbb{N}^3$ such that $\frac{(pqj)-\lambda_{pqj}}{g(pqj)} < \frac{\delta}{2}$ for all $(p, q, j) \geq (u, v, w)$. Now for $\epsilon > 0$ we have

$$\frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \leq (p,q,j) : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$= \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \leq (p,q,j) - \lambda_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$+ \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$\leq \frac{(pqj)-\lambda_{pqj}}{g(pqj)} + \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$\leq \frac{\delta}{2} + \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$

for all $(p, q, j) \geq (u, v, w)$.
Therefore

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \leq (p,q,j) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{(p,q,j) \in \mathbb{N} : \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right| \geq \frac{\delta}{2}\right\}$$
$$\cup \{(1,1,1),(2,2,2),(3,3,3)\cdots,(u,v,w)\}.$$

Hence a triple sequence of Bernstein polynomials of $B_{mnk}(f;x,y,z)$ is rough I-statistically convergent of weight g to $f(x,y,z)$. □

Theorem 3.10. *Let $(B_{mnk}(f;x,y,z))$ be a triple sequence of Bernstein polynomials of real numbers, $g_1, g_2 \in G$ and let $\lambda = (\lambda_{pqj}), \mu = (\mu_{rst})$ be two sequences in η such that $\lambda_{pqj} \leq \mu_{pqj}$ for all $p, q, j \in \mathbb{N}$ if*

$$\liminf_{pqj} \frac{g_1(\lambda_{pqj})}{g_2(\mu_{pqj})} > 0 \tag{3.1}$$

then $(rS(I_\mu))^{g_2} \subset (rS(I_\lambda))^{g_1}$.

Proof. Suppose that $\lambda_{pqj} \leq \mu_{pqj}$ for all $(p, q, j) \in \mathbb{N}^3$ and let (3.1) be satisfied. Now for given $\epsilon > 0$ we have

$$\{(m,n,k) \in J_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}$$
$$\supseteq \{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\},$$

where $I_{pqj} = [(pqj) - \lambda_{pqj} + 1, (pqj)]$ and $J_{pqj} = [(pqj) - \mu_{pqj} + 1, (pqj)]$.
Therefore we have

$$\frac{1}{g_2(\mu_{pqj})} \left|\{(m,n,k) \in J_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$
$$\geq \frac{g_1(\lambda_{pqj})}{g_2(\mu_{pqj})} \frac{1}{g_2(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r+\epsilon\}\right|$$

for all $p,q,j \in \mathbb{N}^3$. So, we get

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_1(\lambda_{pqj})} \left|\left\{(m,n,k) \in I_{pqj} : \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq r+\epsilon\right\}\right| \geq \delta\right\} \subseteq$$
$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_2(\lambda_{pqj})} \left|\left\{(m,n,k) \leq (p,q,j) \in J_{pqj} : \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq r+\epsilon\right\}\right|\right.$$
$$\left. \geq \delta \frac{g_2(\lambda_{pqj})}{g_1(\mu_{pqj})}\right\} \in I.$$

Hence $(rS(I_\mu))^{g_2} \subset (rS(I_\lambda))^{g_1}$. □

Theorem 3.11. *Let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers, if $\{\lambda_{pqj}\} \in \eta$. Then $B_{mnk}(f;x,y,z) \to f(x,y,z)[V,\lambda](rI)^g \Longrightarrow B_{mnk}(f;x,y,z) \to f(x,y,z)(rS(I_\lambda))^g$ and $(I_\lambda)^g \subsetneq [V,\lambda](I)^g$ for every ideal I.*

Proof. Let $\epsilon > 0$ and $B_{mnk}(f;x,y,z) \to f(x,y,z)[V,\lambda](rI)^g$, we have

$$\sum_{(m,n,k)\in I_{pqj}} \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right|$$
$$\geq \sum_{\substack{(m,n,k)\in I_{pqj} \\ \left|B_{mnk}(f;x,y,z)-f(x,y,z)\right|>r+\epsilon}} \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right|$$
$$\geq (r+\epsilon) \cdot \left|\left\{(m,n,k) \in I_{pqj} : \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq r+\epsilon\right\}\right|.$$

Given $\delta > 0$,

$$\frac{1}{g(\lambda_{pqj})} \left|\left\{(m,n,k) \in I_{pqj} : \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq r+\epsilon\right\}\right| \geq \delta$$
$$\Longrightarrow \frac{1}{g(\lambda_{pqj})} \sum_{(m,n,k)\in I_{pqj}} \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq \epsilon\delta.$$

i.e.,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left|\left\{(m,n,k) \in I_{pqj} : \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right| \geq r+\epsilon\right\}\right| \geq \delta\right\}$$
$$\subset \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left\{\sum_{(m,n,k)\in I_{pqj}} \left|B_{mnk}(f;x,y,z) - f(x,y,z)\right|\right\} \geq \epsilon\delta.\right\}$$

since $B_{mnk}(f;x,y,z) \to f(x,y,z)[V,\lambda](rI)^g$ and hence it follows that $B_{mnk}(f;x,y,z) \to f(x,y,z)(rS(I_\lambda))^g$ for proper ideal of I.
Now to prove that $(rS(I_\lambda))^g \subsetneq [V,\lambda](rI)^g$, take a fixed element $V \in I$. A triple sequence Bernstein polynomials defined by

$$B_{mnk}(f:x,y,z) = \begin{cases} (m,n,k) & \text{for } (p,q,j) - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \leq (m,n,k) \leq (p,q,j), (p,q,j) \notin V \\ (m,n,k) & \text{for } (p,q,j) - \lambda_{pqj} + 1 \leq (m,n,k) \leq (p,q,j), (p,q,j) \in V \\ 0 & \text{otherwise} \end{cases}$$

for every $\epsilon > 0\ (0 < \epsilon < 1)$, since
$\frac{1}{g(\lambda_{pqj})} \left|\left\{(m,n,k) \in I_{pqj} : \left|B_{mnk}(f;x,y,z)\right| \geq r+\epsilon\right\}\right| = \frac{\left[\sqrt{g(\lambda_{pqj})}\right]}{g(\lambda_{pqj})} \to 0$ as $p,q,j \to$

∞ and $(p,q,j) \notin V$, for every $\delta > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left|\{(m,n,k) \in I_{pqj} : |B_{mnk}(f;x,y,z) - f(x,y,z)| \geq r + \epsilon\}\right| \geq \delta\right\}$$
$$\subset V \cup \{(1,1,1),(2,2,2),(3,3,3)\cdots,(u,v,w)\} \text{ for some } (u,v,w) \in \mathbb{N}^3.$$

Since I is admissible of weight g, it follows that $B_{mnk}(f;x,y,z) \to 0\,(rS(I_\lambda))^g$. Hence

$$\tfrac{1}{g(\lambda_{pqj})} \sum_{(m,n,k)\in I_{pqj}} |B_{mnk}(f;x,y,z)| \to \infty \text{ as } p,q,j \to \infty$$

i.e $B_{mnk}(f;x,y,z) \not\to 0\,[V,\lambda]\,(rI)^g$, if $V \in I$ is infinite then $B_{mnk}(f;x,y,z) \not\to \theta\,(rS(I_\lambda))^g$. □

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References

[1] S. Aytar, Rough statistical convergence, *Numer. Funct. Anal. Optimiz.* **29(3-4)**(2008) 291-303.

[2] S. Aytar, The rough limit set and the core of a real sequence, *Numer. Funct. Anal. Optimiz.* **29(3-4)**(2008) 283-290.

[3] M. Balcerzak, P. Das, M. Filipczak, and J. Swaczyna, Generalized kinds of density and the associated ideals, *Acta Math. Hungar.* **147(1)**(2015) 97-115.

[4] A. J. Dutta, A. Esi, B.C. Tripathy, Statistically convergent triple sequence spaces defined by Orlicz function, *J. Math. Anal.* **4(2)**(2013) 16-22.

[5] S. Debnath, B. Sarma, B.C. Das, Some generalized triple sequence spaces of real numbers, *J. Nonlinear Anal. Optimiz.* **6(1)** (2015) 71-79.

[6] E. Dündar, C. Çakan, Rough $I-$convergence, *Demonstratio Math.* **XLVII(3)**(2014) 638-651.

[7] E. Dündar, C. Çakan, Rough convergence of double sequences, *Gulf J. Math.* **2(1)**(2014) 45-51.

[8] E. Dündar, On Rough $\mathcal{I}_2$-Convergence of Double Sequences, *Numer. Funct. Anal. Optimiz.* **37(4)**(2016) 480-491.

[9] A. Esi, On some triple almost lacunary sequence spaces defined by Orlicz functions, *Research and Reviews: Discrete Math. Structures* **1(2)**(2014) 16-25.

[10] A. Esi, M. Necdet Catalbas, Almost convergence of triple sequences, *Global J. Math. Anal.* **2(1)**(2014) 6-10.

[11] A. Esi, E. Şavas, On lacunary statistically convergent triple sequences in probabilistic normed space, *Appl. Math. Inf. Sci.* **9(5)**(2015) 2529-2534.

[12] A. Esi, S. Araci, M. Acikgoz, Statistical Convergence of Bernstein Operators, *Appl. Math. Inf. Sci.* **10(6)** (2016) 2083-2086.

[13] A. Esi λ_3-Statistical convergence of triple sequences on probabilistic normed space, *Global J. Math. Anal.* **1(2)** (2013) 29-36

[14] B. Hazarika, S. A.Mohiuddine, Ideal convergence of random variables, *J. Func. Spaces Appl.* **2013**(2013), Article ID 148249, 7 pages.

[15] P. Kostyrko, T. Šalát, W. Wilczyśki, On I-convergence, *Real Analysis Exchange* **26(2)**(2000-2001) 669-686.

[16] P. Malik, M. Maity, On rough statistical convergence of double sequences in normed linear spaces, *Afr. Mat.* **27(1)**(2016) 141-148.

[17] C.P. Niculescu, G.T. Prajitura, Some open problems concerning the convergence of positive series, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* 6(2014) 92-107.

[18] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, *J. Math. Anal. Appl.* **245**(2000) 513-527.

[19] S. K. Pal, D. Chandra, S. Dutta, Rough Ideal Convergence, *Hacettepe J. Math. Stat.* **42(6)**(2013) 633-640.

[20] H.X. Phu, Rough convergence in normed linear spaces, *Numer. Funct. Anal. Optimiz.* **22**(2001) 199-222.

[21] H.X. Phu, Rough continuity of linear operators, *Numer. Funct. Anal. Optimiz.* **23**(2002) 139-146.

[22] H.X. Phu, Rough convergence in infinite dimensional normed spaces, *Numer. Funct. Anal. Optimiz.* **24**(2003) 285-301.

[23] A. Sahiner, M. Gurdal, F.K. Duden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.* **8(2)**(2007) 49-55.

[24] A. Sahiner, B.C. Tripathy, Some I related properties of triple sequences, *Selcuk J. Appl. Math.* **9(2)**(2008) 9-18.

[25] T. Šalát, B. C. Tripathy, M. Ziman, On some properties of I-convergence, *Tatra Mt. Math. Publ.* **28**(2004) 279-286.

[26] E. Savas, A. Esi, Statistical convergence of triple sequences on probabilistic normed space, *Ann. Uni. Craiova, Math. Compu. Sci. Series* **39(2)**(2012) 226-236.

[27] E. Şavas, On I_λ-statistical convergence of weight g, *API Conf. Proc.* 1759 (2016) 1-5.

[28] D. D. Stancu, A Method for Obtaining Polynomials of Bernstein type of two Variables, *The American Math. Monthly*, 70(3)1963) 260-264.

[29] B.C. Tripathy, R. Goswami, On triple difference sequences of real numbers in probabilistic normed spaces, *Proyecciones J. Math.* **33(2)**(2014) 157-174.